George Yang

Generalized Universal Chord Theorem

If a function f is continuous on the closed interval $[a, b]$ and satisfies $f(a) = f(b)$, then given $n \in \mathbb{N}$, there exists $x \in [a, b]$ such that $f(x) = f(x + \frac{b-a}{n})$.

We generalize and prove that if a function f is continuous on the closed interval $[a, b]$ and satisfies $f(a) = f(b)$, then for all $\epsilon < b - a$, there exists $x \in [a, b]$ such that $f(x) = f(x + \epsilon)$.

Proof. Let $0 < \epsilon < b - a$. We have 2 cases: 1) f is the constant function. 2) f is not the constant function. **Case 1** is trivial, as we could pick $x = a$, add ϵ to it to get y, and we'd know that $f(a) = f(a + \epsilon)$.

On to Case 2. We begin with a lemma.

Lemma 1: If $f(a) = f(b)$, f is not a constant function, and f is continuous, we can find a maximum or a minimum on $[a, b]$ that is not $f(a)$ or $f(b)$.

It suffices to show that if we *cannot* find a maximum or a minimum on $[a, b]$ that is not $f(a)$ or $f(b)$, and if $f(a) = f(b)$, then f is a constant function. So, suppose that we *cannot* find a maximum or a minimum on $[a, b]$ that is not $f(a)$ or $f(b)$, and $f(a) = f(b)$. Then, f must achieve both a maximum and a minimum at $f(a)$ and $f(b)$, as this is a closed interval. So, for all $x \in [a, b]$,

$$f(b) = f(a) \leq x \leq f(b) = f(a)$$

So, for all $x \in [a, b]$,

$$x = f(b) = f(a)$$

Thus, f is constant.

So, suppose that f is not the constant function. Then, we can find a maximum or a minimum on $[a, b]$ that is not $f(a)$ or $f(b)$.

Using Lemma 1, we now we have 2 cases:

2.1) f achieves a maximum on (a, b).

2.2) f achieves a minimum on (a, b).

We prove Case 2.1, and Case 2.2 follows by symmetry. Suppose f achieves a maximum on (a, b). Call the pre-image of that maximum value $c \in (a, b)$.

I again split this into two cases:

2.1.a) There exists a closed neighborhood around this maximum that is constant such that the length of this neighborhood is greater than or equal to ϵ.

2.1.b) The converse of Case 2.1.a.

Case 2.1.a:

Suppose there exists a closed neighborhood around this maximum that is constant, such that the length of this neighborhood is greater than or equal to ϵ. Then, there exists $[c, d] \subseteq [a, b]$, such that for all $0 < \eta \leq d - c$, if $|x - y| \leq \eta$, $f(x) = f(y)$, and $\epsilon \leq d - c$. So, since $\epsilon \leq d - c$, we could choose an x, y in this interval such that $|x - y| = \epsilon \leq d - c$. If $\epsilon = d - c$, then we would be done, as we'd have found an y, x such that $f(x) = f(y)$. If $\epsilon < d - c$, by the Archimidean Property, we find an η in between ϵ and $d - c$. So $f(x) = f(y)$. So, we'd have found an x, and a $y = x + \epsilon$ such that $f(x) = f(y) = f(x + \epsilon)$.

Case 2.1.b:

Consider the function

$$g(x) = f(x) - f(x + \epsilon)$$

It suffices to show that there exists an $x \in [a, b]$ such that $g(x) = 0$. We observe that g is a continuous function on $[a, b - \epsilon]$, as $f(x)$ is continuous on $[a, b]$, and $-f(x + \epsilon)$ is continuous on $[a - \epsilon, b - \epsilon]$.

We know f achieves a maximum on (a, b). Call the pre-image of that maximum value $c \in (a, b)$.

Since there may be multiple maxima, use the Archimidean property to find a number, ϵ_2, that satisfies the following properties:

- $\epsilon_2 > d - c$, if there is a closed constant neighborhood $[c, d]$, where $[c, d] \subseteq [a, b]$, such that for all $0 < \eta \leq d - c$, if $|x - y| \leq \eta$, $f(x) = f(y)$, and $\epsilon > d - c$. Note that the $\epsilon > d - c$ is what differentiates this from Case 2.1.a.
- $0 < \epsilon_2 < \epsilon$
- $0 < \epsilon_2 < |k - c|$, where $f(k)$ is any other maximum.

That is, we restrict our attention to the interval such that our maximum at $f(c)$ is greater than both $f(c + \epsilon_2)$ and $f(c - \epsilon_2)$.

Then,

$$f(c) > f(c + \epsilon_2)$$
$$f(c) - f(c + \epsilon_2) > 0$$
$$g(c) > 0$$

Similarly,

$$f(c) > f(c - \epsilon_2)$$
$$f(c) - f(c - \epsilon_2) > 0$$
$$-g(c - \epsilon_2) > 0$$
$$g(c - \epsilon_2) < 0$$

So, we have found two elements, $c, c - \epsilon_2 \in [a, b - \epsilon_2]$, such that g evaluated at these two elements is greater than and less than 0. Thus, by IVT, there exists a $d \in [c - \epsilon_2, c]$ such that $g(d) = 0$. The minimum proof (Case 2.2) is the exact same as the above, but we swap inequality signs. So, we've shown if a function f is continuous on the closed interval $[a, b]$ and satisfies $f(a) = f(b)$, then for all $\epsilon < b - a$, there exists $x \in [a, b]$ such that $f(x) = f(x + \epsilon)$. $\{\epsilon | n \in \mathbb{N}, \epsilon = \frac{b-a}{n}\}$ is a subset of $\{\epsilon | \epsilon \in \mathbb{R}, 0 < \epsilon < b - a\}$, so we've proved the universal chord theorem.

□